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On Ordinary Differential Equations Admitting a Finite Linear Group of Symmetries

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1. INTRODUCTION

The importance of (connected, local) Lie groups of symmetries for the study of differential equations has been recognized since Lie's fundamental work in the last century and has received renewed interest in recent years (cf. Olver [14] for a contemporary account). While Lie's own primary goal may have been the reduction of differential equations to integration problems via reduction of dimension, the construction of a suitable "reduced space" for differential equations admitting a compact linear group of symmetries, and its application to the qualitative theory of such equations, has been the focus of much research in recent times; cf. the monographs by Golubitsky *et al.* [7, 8].

The purpose of this article is to isolate and discuss the algebraic aspect of reduction for differential equations admitting a linear group of symmetries or orbital symmetries which satisfies a (rather weak) additional condition. While the result should be considered well known in the case of a symmetry group, reduction in the presence of a group of orbital symme-

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tries (which preserve solution orbits, but not necessarily their parametrization) seems to have gone unnoticed (or at least unexploited) so far. We discuss a number of examples with finite linear (orbital) symmetry groups to illustrate how the procedure provides quite strong information in some cases, although there is no reduction of dimension. The examples will show that, unlike the connected Lie group case, the consideration of complex symmetry groups may yield useful information even if one is only interested in differential equations in real vector spaces. It will also turn out that orbital symmetries have quite strong consequences, at least when the group is a reflection group.

It is worth mentioning that when dealing with a differential equation of low dimension which exhibits symmetry or orbital symmetry with respect to some group, an experienced observer will often notice (after some experimentation) an appropriate change of variables that renders the equation into a more tractable form. We do not see the reduction procedure discussed herein as competing with such methods; indeed, our view is that the underlying symmetry is the reason such methods work. Our purpose is to present an approach to the study of ordinary differential equations with finite (orbital) symmetry groups which is systematic and thus can be used whether or not an *ad hoc* approach immediately suggests itself.

We start with a few definitions and preliminary results. We will always consider an autonomous ordinary differential equation

(*) $\dot{x} = f(x)$ on an open, connected neighborhood U of 0 in $\mathbb{K}^n$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$) with f analytic on U, and $f \neq 0$.

If another differential equation $\dot{x} = g(x)$ (with analytic g) is given on $V \subseteq \mathbb{K}^m$, and there is a nonempty, open subset $\tilde{U} \subseteq U$ and an analytic map $\Phi: \tilde{U} \to V$ that sends parametrized solutions of $\dot{x} = f(x)$ to parametrized solutions of $\dot{x} = g(x)$, then we call Φ a *solution-preserving map* from $\dot{x} = f(x)$ to $\dot{x} = g(x)$. If $V = U$, $g = f$, and Φ is (locally) invertible then Φ is called a *symmetry* of $\dot{x} = f(x)$. If $V = U$, $g = \mu f$ (with $0 \neq \mu: U \to \mathbb{K}$ analytic), and Φ is (locally) invertible then Φ is called an *orbital symmetry* of $\dot{x} = f(x)$. If μ is not identically 1 then we speak of a *proper orbital symmetry*. (Note that $\dot{x} = f(x)$ and $\dot{x} = \mu(x)f(x)$ have the same solution orbits near every point that is not a zero of μ. Conversely, two equations that have locally the same solution orbits differ only by a scalar factor μ that is necessarily the quotient of two analytic functions; cf. [19].) There is a familiar "infinitesimal" criterion: With notations as above, Φ is solution-preserving from $\dot{x} = f(x)$ to $\dot{x} = g(x)$ if and only if $D\Phi(x)f(x) = g(\Phi(x))$ for all $x \in \tilde{U}$; see, for instance, Olver [14], or [19].

Now let $G \subseteq GL(n, \mathbb{K})$ be a linear group. We call G an (*orbital*) *symmetry group* of $\dot{x} = f(x)$ if every element of G is an (orbital) symmetry of $\dot{x} = f(x)$ (resp. its complexification, if necessary). Occasionally we speak of a *proper orbital symmetry group* if G contains at least one proper orbital symmetry.

The criteria for orbital symmetries can be slightly sharpened for finite G. Recall that a *character* of G is a homomorphism $\chi : G \to \mathbb{C}^*$.

PROPOSITION 1.1. *The finite linear group G is an orbital symmetry group of $\dot{x} = f(x)$ if and only if there is a character χ of G such that $T \circ f \circ T^{-1} = \chi(T)f$ for all $T \in G$. G is a symmetry group of $\dot{x} = f(x)$ if and only if $T \circ f \circ T^{-1} = f$ for all $T \in G$.*

Proof. In view of the discussion above, we only have to prove the necessity of the first condition. If $T \in G$ is an orbital symmetry of $\dot{x} = f(x)$, there is an analytic function μ_T so that $(T \circ f \circ T^{-1})(x) = \mu_T(x)f(x)$ for all x. Because T has finite order and $\boldsymbol{U}$ is connected, μ_T is constant and its value is a $|G|^{\text{th}}$ root of unity, which we call $\chi(T)$. It is easily verified that χ is a character of G. ∎

If φ is analytic in $\boldsymbol{U}^* \subseteq \boldsymbol{U}$ with values in $\mathbb{K}$, the *Lie derivative* $L_f(\varphi)$ of φ with respect to f is defined by $L_f(\varphi) := D\varphi(x)f(x)$. (We call φ a *semi-invariant*, resp. *first integral*, of $\dot{x} = f(x)$ if $L_f(\varphi) = \mu\varphi$ for some analytic $\mu : \boldsymbol{U} \to \mathbb{K}$, resp. $L_f(\varphi) = 0$. The zero level set of a semi-invariant is an invariant set for $\dot{x} = f(x)$, as is any level set of a first integral.)

The following result may be seen as the technical basis for the reduction theorems.

LEMMA 1.2. *Let G be a linear group, and let χ, η be characters of G such that $T \circ f \circ T^{-1} = \chi(T)f$ and $\varphi \circ T^{-1} = \eta(T)\varphi$ for all $T \in G$. Then $L_f(\varphi) \circ T^{-1} = \chi(T)\eta(T)L_f(\varphi)$ for all $T \in G$.*

Proof. For $x \in \boldsymbol{U}^*$, differentiate $\varphi(T^{-1}x) = \eta(T)\varphi(x)$ to obtain $D\varphi(T^{-1}x)T^{-1} = \eta(T)D\varphi(x)$. Now

$$\begin{aligned}
\chi(T)\eta(T)L_f(\varphi)(x) &= \chi(T)\eta(T)D\varphi(x)f(x) \\
&= D\varphi(T^{-1}x)\big(T^{-1}\chi(T)f(x)\big) \\
&= D\varphi(T^{-1}x)f(T^{-1}x) = L_{f(\varphi)}(T^{-1}x). \qquad ∎
\end{aligned}$$

2. REDUCTION IN THE CASE OF A SYMMETRY GROUP

Let G be a subgroup of $GL(n, \mathbb{K})$. Recall that a polynomial $\varphi \in \mathbb{K}[x_1, \ldots, x_n]$ is called an *invariant* of G if $\varphi \circ T^{-1} = \varphi$ for all $T \in G$. Moreover, a polynomial ψ is called a *relative invariant* of G if there is a character η of G so that $\psi \circ T^{-1} = \eta\psi$ for all $T \in G$. The invariants of G form a subalgebra $I(G)$, while the relative invariants corresponding to a fixed character η form an $I(G)$-module we denote by $I_\eta(G)$.

There are linear groups for which $I(G)$ is not a finitely generated $\mathbb{K}$-algebra, but $I(G)$ is finitely generated for large classes of groups, including every group that acts completely reducibly on $\mathbb{K}^n$ (see Springer [17] on these issues) and therefore every finite and every (real) compact group. We state the reduction theorem in a special case:

THEOREM 2.1. *Let G be a linear group such that $I(G)$ has a finite system of generators $\varphi_1, \ldots, \varphi_r$. If f is a polynomial and $T \circ f \circ T^{-1} = f$ for all $T \in G$ then the map*

$$\Phi : \begin{cases} \mathbb{K}^n \to \mathbb{K}^r \\ x \mapsto \begin{pmatrix} \varphi_1 \\ \vdots \\ \varphi_r \end{pmatrix} \end{cases}$$

is solution-preserving from $\dot{x} = f(x)$ to a differential equation $\dot{y} = g(y)$ with g a polynomial.

Proof. We have to verify the existence of a polynomial mapping g such that $D\Phi(x)f(x) = g(\Phi(x))$ for all x. Lemma 1.2 (with $\chi = \eta = 1$) implies $L_f(\varphi) \in I(G)$ for all j, hence there are polynomials γ_j such that $L_f(\varphi) = \gamma_j(\varphi_1, \ldots, \varphi_r)$, $1 \le j \le r$. Thus

$$g = \begin{pmatrix} \gamma_1 \\ \vdots \\ \gamma_r \end{pmatrix}$$

will suffice. ∎

Remark 2.2. (a) If G acts completely reducibly on $\mathbb{K}^n$, then Theorem 2.1 continues to hold with "polynomial" replaced by "analytic." This follows from a theorem of Luna [13], which states that every analytic G-invariant $\mathbb{K}$-valued function σ defined near 0 can be represented as $\sigma = \rho(\varphi_1, \ldots, \varphi_r)$, with ρ a power series that has a nontrivial domain of convergence. (It is this nontriviality that is the difficult part of the result.) For the C^∞ case (with (real) compact G), see [8, 16].

(b) It is not necessary for there to be an algebraically independent set of generators for $I(G)$. The (polynomial) relations between $\varphi_1, \ldots, \varphi_r$ define a variety $Y \subseteq \mathbb{K}^r$ which can be shown to be invariant for $\dot{y} = g(y)$ and contains the image of Φ. Y is sometimes called the discriminant variety of G; see Cox *et al*. [5]. If $\mathbb{K} = \mathbb{C}$ and G is finite, then points of Y are in one-to-one correspondence with orbits of G. If $\mathbb{K} = \mathbb{R}$ and G is compact then there is a semi-algebraic subset $\tilde{Y}$ of Y whose points are in one-to-one correspondence with orbits of G; see Schwarz [16] for the inequalities defining $\tilde{Y}$.

(c) For real compact groups, the structure and geometry of the aforementioned *orbit space* (the semialgebraic variety $\tilde{Y}$) has been discussed both in general and for many specific groups; see Gaeta [6], Abud and Sartori [1, 2], Chossat [4], and Jaric *et al*. [11]. It should be noted that another approach to characterize strata in the orbit space uses isotropy subgroups, cf. [6, 1, 2, 8].

(d) Concerning the question of finding a system of generators for $I(G)$, G finite, see Cox *et al*. [5, Chap. 7] and Sturmfels [18].

(e) For a given differential equation with polynomial right hand side, the problem of finding linear (orbital) symmetries leads to a system of polynomial equations for the matrix entries (after some coordinate system is fixed). In principle, this is accessible with Gröbner basis methods (see Cox *et al*. [5]).

The following examples will illustrate how Theorem 2.1 may help in analyzing or solving a differential equation. Note that most of them are taken from problems arising in physics or mathematical biology, and are therefore of some "practical" interest.

EXAMPLE 2.3. Kasner's equation (cf. [12])

$$\begin{aligned} \dot{x}_1 &= x_2 x_3 - x_1^2 \\ \dot{x}_2 &= x_3 x_1 - x_2^2 \qquad \left(\text{briefly, } \dot{x} = f(x) \text{ in } \mathbb{R}^3\right) \\ \dot{x}_3 &= x_1 x_2 - x_3^2 \end{aligned}$$

describes Einstein's gravitation equations in a special case. This equation has S_3 (the group of linear maps defined by permutations of the standard basis) as a symmetry group. The invariant algebra $I(S_3)$ is generated by the elementary symmetric polynomials $\varphi_1(x) = x_1 + x_2 + x_3$, $\varphi_2(x) = x_1 x_2 + x_2 x_3 + x_3 x_1$, $\varphi_3(x) = x_1 x_2 x_3$. Therefore

$$\Phi = \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \end{pmatrix}$$

is solution-preserving from $\dot{x} = f(x)$ to an equation $\dot{y} = g(y)$ in $\mathbb{R}^3$ that turns out to be easily solved by elementary functions; see [12]. (It should be noted that $\dot{x} = f(x)$ admits a two-dimensional abelian linear group of orbital symmetries, generated by dilations and rotations about

$$\mathbb{R} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

Therefore the solution of $\dot{x} = f(x)$ can be reduced to integration problems á la Lie. We include this example for historical reasons; it shows that for specific examples, the reduction technique has been around for some time.)

EXAMPLE 2.4. The real equation

$$\begin{aligned}\dot{x}_1 &= \alpha x_1 + \beta\left(x_1^4, x_1x_2, x_2^4\right)x_2^3\\ \dot{x}_2 &= \alpha x_2 - \beta\left(x_1^4, x_1x_2, x_2^4\right)x_1^3\end{aligned}$$

($\dot{x} = f(x)$ in $\mathbb{R}^2$)—with α a real number and $\beta : \mathbb{R}^3 \to \mathbb{R}$ a polynomial—admits the complex symmetry group G generated by $T = \operatorname{diag}(i, -i)$. The algebra $I(G)$ is generated by $\varphi_1(x) = x_1^4$, $\varphi_2(x) = x_1x_2$, and $\varphi_3(x) = x_2^4$. Now

$$\begin{aligned}L_f(\varphi_1) &= 4\alpha x_1^4 + 4\beta\cdot(x_1x_2)^3 = 4\alpha\varphi_1 + 4\beta\varphi_2^3,\\ L_f(\varphi_2) &= 2\alpha x_1x_2 + \beta\cdot\left(x_2^4 - x_1^4\right) = 2\alpha\varphi_2 + \beta\cdot(\varphi_3 - \varphi_1),\\ L_f(\varphi_3) &= 4\alpha x_2^4 - 4\beta\cdot(x_1x_2)^3 = 4\alpha\varphi_3 - 4\beta\varphi_2^3.\end{aligned}$$

Therefore

$$\Phi = \begin{pmatrix}\varphi_1\\ \varphi_2\\ \varphi_3\end{pmatrix}$$

is solution-preserving from $\dot{x} = f(x)$ into

$$\begin{aligned}\dot{y}_1 &= 4\alpha y_1 + 4\beta(y_1, y_2, y_3)y_2^3\\ \dot{y}_2 &= 2\alpha y_2 + \beta(y_1, y_2, y_3)(y_3 - y_1) \qquad \left(\dot{y} = g(y) \text{ in } \mathbb{R}^3\right)\\ \dot{y}_3 &= 4\alpha y_3 - 4\beta(y_1, y_2, y_3)y_2^3.\end{aligned}$$

(The variety $Y \subseteq \mathbb{R}^3$ (see Remark 2.2(b)) is defined as the zero set of $y_1y_3 - y_2^4$, since $\varphi_1\varphi_3 - \varphi_2^4 = 0$ is essentially the only relation between $\varphi_1, \varphi_2, \varphi_3$.) While $\dot{x} = g(x)$ does not appear to be any easier to solve than the original equation, we can still easily obtain nontrivial information from it: Obviously, $(y_1 + y_3)' = 4\alpha(y_1 + y_3)$, and pulling this back yields $L_f(\varphi_1 + \varphi_3) = 4\alpha(\varphi_1 + \varphi_3)$. Thus $\sigma := \varphi_1 + \varphi_3 : \mathbb{R}^2 \to \mathbb{R}$ is a solution-preserving map from $\dot{x} = f(x)$ into the one-dimensional equation $\dot{y} = 4\alpha y$, and this yields enough information to determine the qualitative behavior. As can be seen here, complex symmetries may provide interesting information about real systems.

EXAMPLE 2.5. Equations of the type

$$\begin{aligned}\dot{x}_1 &= x_1(\mu x_1 + \nu(x_2 + x_3))\\ \dot{x}_2 &= x_2(\mu x_2 + \nu(x_3 + x_1)) \qquad (\dot{x} = f(x) \text{ in } \mathbb{R}^3, \text{with real } \mu, \nu)\\ \dot{x}_3 &= x_3(\mu x_3 + \nu(x_1 + x_2))\end{aligned}$$

are of some importance in selection models of mathematical biology; see Hofbauer and Sigmund [10]. From S_3-invariance it follows that

$$\Phi = \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \end{pmatrix}$$

(with the elementary symmetric polynomials) is solution-preserving to an equation $\dot{y} = g(y)$ in $\mathbb{R}^3$. Computation shows

$$g(y) = \begin{pmatrix} \mu y_1^2 \\ (\mu + \nu) y_1 y_2 \\ (\mu + 2\nu) y_1 y_3 \end{pmatrix} + (\nu - \mu) \begin{pmatrix} 2y_2 \\ 3y_3 \\ 0 \end{pmatrix} = y_1(h(y) + k(y)),$$

where

$$h(y) := \begin{pmatrix} \mu y_1 \\ (\mu + \nu) y_2 \\ (\mu + 2\nu) y_3 \end{pmatrix} \qquad \text{and} \qquad k(y) := \frac{(\nu - \mu)}{y_1} \begin{pmatrix} 2y_2 \\ 3y_3 \\ 0 \end{pmatrix}.$$

For the Lie bracket of these two vector fields, we compute

$$[h, k](y) = Dk(y)h(y) - Dh(y)k(y) = (\nu - \mu)k(y).$$

From this commutator relation it follows that the general solution of $\dot{y} = h(y) + k(y)$ can be expressed via the general solutions of $\dot{y} = h(y)$ and $\dot{y} = k(y)$: Denote by $H(t, z)$ the solution of $\dot{y} = h(y)$, $y(0) = z$, and by $K(t, z)$ the solution of $\dot{y} = k(y)$, $y(0) = z$. Then the solution of $\dot{y} = h(y) + k(y)$, $y(0) = z$, is given by $H(t, K((1/(\nu - \mu)))(e^{(\nu-\mu)t} - 1), z))$. (We assume $\mu \neq \nu$; the case $\mu = \nu$ allows easy integration of $\dot{y} = g(y)$.) The proof of this formula follows as in [20]. Now

$$H(t, z) = \begin{pmatrix} z_1 \exp(\mu t) \\ z_2 \exp((\mu + \nu)t) \\ z_3 \exp((\mu + 2\nu)t) \end{pmatrix},$$

while $K(t, z)$ is found from the solution of

$$\dot{y} = (\nu - \mu)\begin{pmatrix} 2y_2 \\ 3y_3 \\ 0 \end{pmatrix} \qquad \left(\text{which is } \begin{pmatrix} z_1 + (\nu-\mu)tz_2 + \frac{1}{2}(\nu-\mu)^2 t^2 z_3 \\ z_2 + (\nu-\mu)tz_3 \\ z_3 \end{pmatrix}\right)$$

by replacing t with the solution $\tau(t)$ to $\dot{\tau} = 1/(z_1 + (\nu-\mu)\tau z_2 + \frac{1}{2}(\nu - \mu)^2\tau^2 z_3)$, $\tau(0) = 0$. (Thus τ can be explicitly determined from the equation $z_1\tau + \frac{1}{2}(\nu - \mu)z_2\tau^2 + \frac{1}{6}(\nu - \mu)^2 z_3\tau^3 = t$ using Cardano's formula.) Putting pieces together, we see that the solution of $\dot{y} = (1/y_1)g(y)$ can be determined explicitly, and so can the solution of $\dot{x} = (1/\varphi_1)f(x)$. (Note that Φ can be explicitly inverted, using Cardano's formula once more.) In the positive orthant (which is the interesting region in applications) this equation and $\dot{x} = f(x)$ have the same solution orbits, and thus the analysis of $\dot{x} = f(x)$ can be carried out by elementary means. The reduction from $\dot{x} = f(x)$ to $\dot{y} = g(y)$ is a quite important step in this example, because it yields an easy-to-recognize decomposition of $(1/y_1)g(y)$ into a sum of vector fields with a well-behaved commutator relation. Moreover, it can be checked that there is no corresponding additive decomposition for $\dot{x} = f(x)$ itself.

EXAMPLE 2.6. The equation

$$\begin{aligned} \dot{x}_1 &= x_2 x_3 \\ \dot{x}_2 &= x_3 x_1 \qquad (\dot{x} = f(x) \text{ in } \mathbb{R}^3) \\ \dot{x}_3 &= -x_1 x_2 \end{aligned}$$

comes from the Euler equation for the (asymmetric) spinning top in the absence of external forces by appropriate scaling of the variables. This equation admits the complex linear symmetries

$$T_1 x := \begin{pmatrix} x_2 \\ x_1 \\ x_3 \end{pmatrix}, \qquad T_2 x := \begin{pmatrix} ix_3 \\ x_2 \\ -ix_1 \end{pmatrix}, \qquad \text{and} \qquad T_3 x := \begin{pmatrix} -x_1 \\ -x_2 \\ x_3 \end{pmatrix},$$

which generate a group of order 12. (It can be shown that no other linear symmetries exist for $\dot{x} = f(x)$.)

G has no nontrivial linear invariant, and (up to scalar multiples) one quadratic invariant $\psi_1(x) = x_1^2 + x_2^2 - x_3^2$. Now $L_f(\psi_1) = 6\psi_2$, with $\psi_2(x) := x_1x_2x_3$, $L_f(\psi_2) = \psi_3$, with $\psi_3(x) := -x_1^2x_2^2 + x_2^2x_3^2 + x_3^2x_1^2$, and

$L_f(\psi_3)(x) = -4x_1x_2x_3(x_1^2 + x_2^2 - x_3^2) = -4\psi_1\psi_2$. Thus

$$\Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix}$$

is solution-preserving from $\dot{x} = f(x)$ into

$$\begin{aligned} \dot{y}_1 &= 6y_2 \\ \dot{y}_2 &= y_3 \qquad \text{in } \mathbb{R}^3, \\ \dot{y}_3 &= -4y_1y_2 \end{aligned}$$

which is equivalent to the third order equation $\dddot{y} = -4y\dot{y}$. Integrating yields $\ddot{y} = -2y^2 + \gamma$ ($\gamma \in \mathbb{R}$), which is a second-order equation for a Weierstrass $\wp$-function. Thus the reduction procedure provides a natural route to the (well-known) fact that Euler's equations can be solved with the help of elliptic functions.

The invariants ψ_1, ψ_2, and ψ_3 do indeed generate $I(G)$ (this follows most easily from the fact that G is conjugate to the tetrahedral group in $GL(3, \mathbb{C})$ and using Jaric *et al.* [11]). It may be worth noting, however, that in this example we did not start with a system of generators for $I(G)$, but only with an element of smallest degree. Thus it is not necessary to have a complete system of generators at the beginning.

To close this section we note another consequence of Lemma 1.2 that is occasionally useful in determining invariant sets. (For sake of simplicity we do not consider the most general situation.)

PROPOSITION 2.7. *Let $G \subseteq GL(n, \mathbb{K})$ and η a character of G such that the $I(G)$-module $I_\eta(G)$ has a finite set of generators $\rho_1, \dots, \rho_s$. If f is a polynomial and G is a symmetry group for $\dot{x} = f(x)$ then the set Z of common zeros of $\rho_1, \dots, \rho_s$ is an invariant set for $\dot{x} = f(x)$.*

Proof. According to Lemma 1.2, $L_f(\rho_i) \in I_\eta(G)$ for $1 \le i \le s$. Therefore there exist $\mu_{ij} \in I(G)$ such that $L_f(\rho_i) = \sum \mu_{ij}\rho_j$ $(1 \le i \le s)$. The invariance of Z follows from [19]. ∎

EXAMPLE 2.8. For $S_3 \subseteq GL(3, \mathbb{R})$ consider the character η that attains the value -1 on the reflections. It is known that the discriminant $\rho(x) = (x_1 - x_2)(x_2 - x_3)(x_3 - x_1)$ generates the $I(S_3)$-module $I_\eta(S_3)$; cf. [17]. Therefore the planes $x_i - x_j = 0$ $(i \ne j)$ are invariant sets for every S_3-symmetric equation. For the equations from Example 2.5, one has additional semi-invariants x_1, x_2, x_3; and the method developed in Gröbner and Knapp [9] yields a number of first integrals, e.g., $\sigma(x) = (x_2 - x_3)^{-(\mu+2\nu)}(x_2x_3)^{(\mu+\nu)}x_1^{-\nu}$.

3. REDUCTION IN THE CASE OF AN ORBITAL SYMMETRY GROUP

Intuitively, a (proper) orbital symmetry group should be less restrictive for a differential equation than a symmetry group. Indeed, reduction has a more complicated look here. Once more we formulate the theorem only for the case of a polynomial differential equation.

THEOREM 3.1. *Let G be linear group and $\chi : G \to \mathbb{C}^*$ a character of order $s+1>1$. Suppose that $I(G)$ has a finite system $\varphi_1, \dots, \varphi_r$ of generators and further that every $I(G)$-module $I_{\chi^k}(G)$ $(1 \le k \le s)$ is finitely generated by $\psi_{k,1}, \dots, \psi_{k,m_k}$. Let $p_k = \sum_{j=1}^{k} m_k$, $1 \le k \le s$. If $T \circ f \circ T^{-1} = \chi(T) f$ for every $T \in G$, then the map*

$$\Phi : \begin{cases} \mathbb{K}^n \to \mathbb{K}^R \qquad (\text{with } R = r + m_1 + \cdots + m_s) \\ x \mapsto \big(\varphi_1(x), \dots, \varphi_r(x), \psi_{1,1}(x), \dots, \psi_{1,m_1}(x), \dots, \\ \psi_{s,1}(x), \dots, \psi_{s,m_s}(x)\big)^t \end{cases}$$

is solution-preserving from the polynomial differential equation $\dot{x} = f(x)$ in $\mathbb{K}^n$ to a polynomial differential equation $\dot{y} = g(y)$ in $\mathbb{K}^R$ which is of the following specific type,

$$\dot{y}_i = \sum_{j=r+1}^{r+p_1} \mu_{ij}^{(0)}(y_1, \dots, y_r) y_j \qquad (1 \le i \le r)$$

$$\dot{y}_i = \sum_{j=r+p_1+1}^{r+p_2} \mu_{ij}^{(1)}(y_1, \dots, y_r) y_j \qquad (r+1 \le i \le r+p_1)$$

$$\vdots$$

$$\dot{y}_i = \sum_{j=r+p_{s-1}+1}^{r+p_s} \mu_{ij}^{(s-1)}(y_1, \dots, y_r) y_j \qquad (r+p_{s-2}+1 \le i \le r+p_{s-1})$$

$$\dot{y}_i = \nu_i(y_1, \dots, y_r) \qquad (r+p_{s-1}+1 \le i \le r+p_s)$$

with all the $\mu_{ij}^{(k)}$ and ν_i polynomials.

Proof. By Lemma 1.2 we have $L_f(\varphi_i) \in I_\chi(G)$ $(1 \le i \le r)$, and $L_f(\psi_{k,j}) \in I_{\chi^{k+1}}(G)$ $(1 \le j \le m_k)$ for all k, where $I_1(G) := I(G)$. ∎

Remark 3.2. (a) The image of Φ is contained in the subvariety Y of $\mathbb{K}^R$ defined by all the relations between the φ_i and $\psi_{k,j}$; and again Y is invariant for $\dot{y} = g(y)$.

(b) As there seems to be no counterpart of Luna's theorem [13] available in the literature for this situation, generalization to the analytic case has to be investigated individually for every group.

(c) Theorem 3.1 turns out to be most useful in the case that every module $I_{\chi^k}(G)$ $(1 \le k \le s)$ is generated by one element. The underlying reason is that the image equation $\dot{y} = g(y)$ starts with a "block"

$$\begin{aligned} \dot{y}_1 &= y_{r+1}\,\mu_1(y_1, \dots, y_r) \\ &\vdots \\ \dot{y}_r &= y_{r+1}\,\mu_r(y_1, \dots, y_r) \end{aligned}$$

and information about solution orbits of the differential equation

$$\begin{aligned} \dot{y}_1 &= \mu_1(y_1, \dots, y_r) \\ &\vdots \\ \dot{y}_r &= \mu_r(y_1, \dots, y_r) \end{aligned}$$

in $\mathbb{K}^r$ can be carried over to $\dot{y} = g(y)$. (For instance, a first integral of the latter will also be a first integral of $\dot{y} = g(y)$.) In the case of finite groups it is known that reflection groups have the desired property (cf. Springer [17, Theorem 4.3.4]). The following examples will therefore involve orbital symmetry groups that are generated by reflections.

EXAMPLE 3.3. The equation

$$\begin{aligned} \dot{x}_1 &= x_1(x_2 - x_3) \\ \dot{x}_2 &= x_2(x_3 - x_1) \qquad (\dot{x} = f(x) \text{ in } \mathbb{R}^3) \\ \dot{x}_3 &= x_3(x_1 - x_2) \end{aligned}$$

is of some significance for a mathematical model in biology; cf. Hofbauer and Sigmund [10, Sect. 16.5]. (It may also be viewed as describing the population dynamics of three species that cyclically prey upon each other.) For this equation we have S_3 as a proper orbital symmetry group, with χ the only nontrivial character of S_3, and $I_\chi(S_3)$ generated by the discriminant: cf. Example 2.8.

The image equation $\dot{y} = g(y)$ turns out to be

$$\begin{aligned} \dot{y}_1 &= 0 \\ \dot{y}_2 &= y_4 \\ \dot{y}_3 &= 0 \\ \dot{y}_4 &= -6y_2^2 + y_1^2 y_2 + 9y_1 y_3. \end{aligned}$$

Here the first integrals y_1 and y_3 are immediate, and pulling these back yields the first integrals $\rho_1(x) = x_1 + x_2 + x_3$ and $\rho_2(x) = x_1 x_2 x_3$ for $\dot{x} = f(x)$. These are already known from [10] (and ρ_1, at least, is obvious by inspection). Returning to the image equation, for $y_1 = \gamma_1$, $y_3 = \gamma_3$ constant we obtain the two-dimensional Hamiltonian system

$$\begin{aligned}\dot{y}_2 &= y_4\\ \dot{y}_4 &= -6y_2^2 + y_1^2 y_2 + 9y_1 y_3,\end{aligned}$$

with first integral $\frac{1}{2}y_4^2 + 2y_2^3 - \frac{1}{2}\gamma_1^2 y_2 - 9\gamma_1\gamma_3 y_2$. Thus we have $(\dot{y}_2)^2 = -4y_2^3 + \gamma_1^2 y_2^2 + 18\gamma_1\gamma_3 y_2 + \delta$ (with δ determined by initial conditions), and this is essentially the differential equation of a Weierstrass $\wp$-function (unless the right hand side has multiple roots, in which case the solution is elementary). Thus in addition to the known first integrals, we add the new piece of information that $\dot{x} = f(x)$ can be solved by employing elementary and elliptic functions.

EXAMPLE 3.4. (a) Let G be the group (of order 4) generated by $T_1 = \mathrm{diag}(1, -1)$ and $T_2 = \mathrm{diag}(-1, 1)$, and let χ be the character of G with value -1 on T_1 and T_2. The algebra $I(G)$ is generated by $\varphi_1(x) = x_1^2$ and $\varphi_2(x) = x_2^2$, and the module $I_\chi(G)$ is generated by $\psi(x) = x_1 x_2$. Consider the differential equation

$$\begin{aligned}\dot{x}_1 &= x_2\, p\left(x_1^2, x_2^2\right)\\ \dot{x}_2 &= x_1\, q\left(x_1^2, x_2^2\right)\end{aligned}$$

($\dot{x} = f(x)$ in $\mathbb{R}^2$) where $p, q : \mathbb{R}^2 \to \mathbb{R}$ are polynomials. Then G is an orbital symmetry group for this equation, with character χ. The map $\Phi = (\varphi_1, \varphi_2, \psi)^t$ is solution-preserving from $\dot{x} = f(x)$ into

$$\begin{aligned}\dot{y}_1 &= 2y_3 p(y_1, y_2)\\ \dot{y}_2 &= 2y_3 q(y_1, y_2)\\ \dot{y}_3 &= y_2 p(y_1, y_2) + y_1 q(y_1, y_2).\end{aligned}$$

(Here the variety Y is given by $y_1 y_2 - y_3^2 = 0$.) We are primarily interested in the first two lines. In particular, if p and q are such that determining a first integral σ of the equation

$$\begin{aligned}\dot{y}_1 &= p(y_1, y_2)\\ \dot{y}_2 &= q(y_1, y_2)\end{aligned}$$

is elementary, then $\rho(x) := \sigma(x_1^2, x_2^2)$ will be an elementary first integral of $\dot{x} = f(x)$. Specific examples include the cases where p and q are given by a Hamiltonian function $(p(x_1, x_2) = \partial_2 h(x_1, x_2),\ \ q(x_1, x_2) = -\partial_1 h(x_1, x_2))$, or where p and q are affine $(p(x_1, x_2) = \mu + \alpha x_1 + \beta x_2$, $q(x_1, x_2) = \nu + \gamma x_1 + \delta x_2)$; the latter case includes every polynomial differential equation of degree ≤ 3 satisfying $T \circ f \circ T^{-1} = \chi(T) f$ for all $T \in G$.

One may view the reduction from a slightly different perspective. For instance, in part (a) one has

$$\frac{d}{dt}\left(x_1^2\right) = 2x_1 x_2 p\left(x_1^2, x_2^2\right)$$
$$\frac{d}{dt}\left(x_2^2\right) = 2x_1 x_2 q\left(x_1^2, x_2^2\right),$$

and thus the mapping $x \mapsto (\varphi_1, \varphi_2)$ is locally orbit-preserving from $\dot{x} = f(x)$ to

$$\dot{y}_1 = p(y_1, y_2)$$
$$\dot{y}_2 = q(y_1, y_2)$$

So it can be said that the reduction consists of a local coordinate transformation, together with a change in time scale. As noted in the Introduction, an experienced observer may, of course, notice this directly. The point we are making here, however, is that the symmetry group gives the underlying reason why this procedure works, and Theorem 3.1 provides a systematic approach.

(b) Let $\zeta = \exp(2\pi i/3)$, $T = \mathrm{diag}(\zeta, 1)$, and G be the group generated by T. Denote by χ the character that has value ζ on T. The algebra $I(G)$ is generated by $\varphi_1(x) = x_1^3$ and $\varphi_2(x) = x_2$, while the modules $I_\chi(G)$ and $I_{\chi^2}(G)$ are generated by $\psi_1(x) = x_1^2$ and $\psi_2(x) = x_1$, respectively. Consider the differential equation

$$\dot{x}_1 = p\left(x_1^3, x_2\right)$$
$$\dot{x}_2 = x_1^2 q\left(x_1^3, x_2\right)$$

$(\dot{x} = f(x)$ in $\mathbb{R}^2)$ where $p, q : \mathbb{R}^2 \to \mathbb{R}$ are polynomials. Then G is an orbital symmetry group for this equation, with character χ. The map $\Phi = (\varphi_1, \varphi_2, \psi_1, \psi_2)^t$ is solution-preserving from $\dot{x} = f(x)$ into

$$\dot{y}_1 = 3y_3 p(y_1, y_2)$$
$$\dot{y}_2 = y_3 q(y_1, y_2)$$
$$\dot{y}_3 = 2y_4 p(y_1, y_2)$$
$$\dot{y}_4 = p(y_1, y_2).$$

(The variety Y is given by $y_1 - y_4^3 = 0$, $y_3 - y_4^2 = 0$.) Again, the first two lines are of principal interest. If p and q are such that determining a first integral σ of the equation

$$\dot{y}_1 = 3p(y_1, y_2)$$
$$\dot{y}_2 = q(y_1, y_2)$$

is elementary, then $\rho(x) := \sigma(x_1^3, x_2)$ will be an elementary first integral of $\dot{x} = f(x)$. Examples include the Hamiltonian case, and the case where $p(x_1, x_2) = \alpha x_1 + u(x_2)$ for some polynomial u and $q(x_1, x_2) = \beta + \gamma x_2$, for this reduces to two one-dimensional linear equations. (The latter case includes every polynomial differential equation of degree ≤ 3 satisfying $T \circ f \circ T^{-1} = \chi(T) f$ for all $T \in G$.) This example provides another indication that complex orbital symmetry groups carry useful information about real equations.

EXAMPLE 3.5. The existence of orbital symmetries forces certain qualitative properties on the differential equation, which in some instances can be recovered from the reduction procedure. We illustrate this assertion by considering the following class of "reversible" systems in $\mathbb{K}^n$, $n \geq 2$,

$$\dot{x} = f(x) := \begin{bmatrix} O & O & \\ O & \begin{matrix} 0 & \alpha \\ \beta & 0 \end{matrix} \end{bmatrix} \cdot x + \text{higher order terms},$$

with f satisfying $T \circ f \circ T^{-1} = -f$ for $T := \operatorname{diag}(1, \dots, 1, -1)$. (Note that the linear part $B = Df(0)$ satisfies $T \circ B \circ T^{-1} = -B$. Moreover, up to a suitable change of coordinates B is the most general linear map admissible.) In the following we will assume $\alpha \neq 0$. The invariant algebra of the group $G = \{T, \mathrm{id}\}$ is generated by $\varphi_1(x) = x_1, \dots, \varphi_{n-1}(x) = x_{n-1}$ and $\varphi_n(x) = x_n^2$, and the module $I_\chi(G)$ is generated by $\psi(x) = x_n$. (Here χ denotes the character mapping T to -1.) By Theorem 3.1 the image equation $\dot{y} = g(y)$ in $\mathbb{K}^{n+1}$ is of the type

$$\begin{aligned} \dot{y}_1 &= y_{n+1}\, \mu_1(y_1, \dots, y_n) \\ &\vdots \\ \dot{y}_n &= y_{n+1}\, \mu_n(y_1, \dots, y_n) \\ \dot{y}_{n+1} &= \nu(y_1, \dots, y_n). \end{aligned}$$

(As follows from Remark 3.2(c), this is also true for analytic f.) Since $L_f(\varphi_{n-1}(x)) = \alpha x_n + \text{h.o.t.} = \psi(x)(\alpha + \text{h.o.t.})$ we have $\mu_{n-1}(0, \dots, 0) = \alpha$

$\neq 0$. Therefore 0 is not a stationary point of

$$\begin{aligned} \dot{y}_1 &= \mu_1(y_1, \ldots, y_n) \\ &\vdots \\ \dot{y}_n &= \mu_n(y_1, \ldots, y_n) \end{aligned} \qquad (\dot{y} = h(y)),$$

and therefore $\dot{y} = h(y)$ has $n - 1$ independent first integrals $\sigma_1, \ldots, \sigma_{n-1}$ near 0. These (considered as functions of $y_1, \ldots, y_{n+1}$) are also first integrals of $\dot{y} = g(y)$, and pulling back yields the first integrals $\rho_i(x) = \sigma_i(x_1, \ldots, x_{n-1}, x_n^2)$ for $\dot{x} = f(x)$ (which, again, are easily checked to be independent).

Therefore $\dot{x} = f(x)$ has $n - 1$ independent first integrals in a neighborhood of the stationary point 0, a highly unusual property.

Let us look at two special cases. (Both are reversible systems in $\mathbb{R}^2$.)

$$\dot{x} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \cdot x + \text{h.o.t.} = f(x) \qquad \text{in } U \subseteq \mathbb{R}^2. \tag{a}$$

The image equation here is

$$\begin{aligned} \dot{y}_1 &= y_3(1 + \text{h.o.t.}) \\ \dot{y}_2 &= y_3(-2y_1 + \text{h.o.t.}) \qquad (\dot{y} = g(y)). \\ \dot{y}_3 &= \cdots. \end{aligned}$$

We obtain a first integral $\sigma(y) = y_2 + y_1^2 + \text{h.o.t.}$ for $\dot{y} = g(y)$ and thus a first integral $\rho(x) = x_1^2 + x_2^2 + \text{h.o.t.}$ for $\dot{x} = f(x)$. Since ρ has a local minimum at $0 \in \mathbb{R}^2$, the (local) level sets of ρ near 0 are closed curves by the classification of critical points in $\mathbb{R}^2$. Therefore the stationary point 0 is a *center* for $\dot{x} = f(x)$. Of course it is well known that a reversible system with linear part as stipulated above has a center at 0 (cf. Sansone and Conti [15] for another proof), but here we have an elementary "algebraic" proof of this fact.

$$\dot{x} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \cdot x + \begin{pmatrix} \beta_3 x_1 x_2 \\ \beta_1 x_1^2 + \beta_2 x_2^2 \end{pmatrix} + \text{h.o.t.} \qquad \text{in } U \subseteq \mathbb{R}^2. \tag{b}$$

(The quadratic term here is the most general compatible with reversibility. We will assume $\beta_1 \neq 0$.) The image equation $\dot{y} = g(y)$ is

$$\begin{aligned} \dot{y}_1 &= y_3(1 + \beta_3 y_1 + \text{h.o.t.}) \\ \dot{y}_2 &= y_3\big(2\beta_2 y_2 + 2\beta_1 y_1^2 + \text{h.o.t.}\big) \qquad (\dot{y} = g(y)). \\ \dot{y}_3 &= \cdots. \end{aligned}$$

Here we get a first integral $\sigma(y) = y_2 - 2\beta_2 y_1 y_2 - (2/3)\beta_1 y_1^3 +$ (irrelevant terms) for $\dot{y} = g(y)$, hence a first integral $\rho(x) = x_2^2 - 2\beta_2 x_1 x_2^2 - (2/3)\beta_1 x_1^3 +$ (irrelevant terms) for $\dot{x} = f(x)$. ("Irrelevance" here is with respect to the level set $\rho(x) = 0$; see Bruno [3, Chap. 1, Sect. 2] on the Newton polygon behind this argument.) For $\rho(x) = 0$ we have

$$\begin{aligned} 0 &= x_2^2(1 - 2\beta_2 x_1) - \tfrac{2}{3}\beta_1 x_1^3 + \cdots, \\ 0 &= x_2^2 - \tfrac{2}{3}\beta_1 x_1^3 (1 - 2\beta_2 x_1)^{-1} + \cdots \\ &= x_2^2 - \tfrac{2}{3}\beta_1 x_1^3 + \cdots, \end{aligned}$$

and the stationary point turns out to be a cusp. Again it is worth noting that an *a priori* determination of the topological type of the stationary point 0 is not an easy problem. The algebraic reduction formalism, however, offers easy access to its solution.

The specific results in the two-dimensional cases can also be obtained using normal forms, although the arguments given above use less machinery. (For instance, in (a) one has to use the nontrivial theorem that reversibility is preserved by a suitable transformation into normal form, and that Bruno's "Condition A" (see [3]) guarantees convergence of such a transformation. In (b), one must argue similarly using the Belitskii normal form associated with the given nilpotent linear part.) In higher dimensions it seems that the normal form machinery is less suitable than the approach chosen here.

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